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THE RELATION OF SPACE AND GEOMETRY TO EXPERIENCE

IV

The Extension of Space Beyond the Bounds of Experience*

IN our last lecture, we arrived at a definition of a general triad of parallel or concurrent lines, instead of what we were looking for—a definition of a general triad of concurrent, not of parallel lines. We wish now to ascertain to what extent the definition we obtained may take the place of that for which we searched in the definition of a point as a class of concurrent lines. Now, it is one of the commonplaces of elementary geometry that parallel lines share many of the most important properties of intersecting lines. Two coplanar lines, for example, must either be parallel or intersect; three planes not all possessing a line in common may intersect either in three concurrent or in three parallel lines; and so on indefinitely. We have just given a definition of a generalized point in terms of the relation of concurrence among three lines: we said that a generalized point was a class a of lines, in the sense in which lines have been already defined, such that there are

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* Professor G. A. Pfeiffer of Columbia has called to my attention the fact that the definition of a point as a set of convex solids each intersecting each and excluding no solid intersecting all, is too inclusive. It includes the set of all convex solids intersecting all the sides of a given triangle. The definition should be amended to read: A point is a set of convex solids each intersecting each, and such that if every solid of the set intersects some convex solid of a set σ, then some solid of σ belongs to the set.

two intersecting lines, *l* and *m*, which are so related to *a* that *a* is the class made up (1) of all the lines *x* which are of such a nature that *l*, *m*, and *x* are concurrent, and (2) of the lines *l* and *m* themselves. The analogy between intersection or concurrence, on the one hand, and parallelism, on the other, suggests to us an interesting logical experiment: let us substitute the word "coplanar" (i. e., parallel or intersecting) for "intersecting" in the above definition, and the phrase "concurrent or parallel" for the word "concurrent," and let us see what we shall obtain as the result. The amended definition will now read as follows: A generalized point is a class *a* of lines, in the sense in which lines have been already defined—that is, to all intents and purposes, a class of all the linear segments which span that region of space wherein convex solids are experienced to intersect—such that two such lines, *l* and *m*, which lie in the same plane can be assigned which determine *a* in such a way that it is made up (1) of all lines *x* such that *l*, *m*, and *x* are either concurrent or parallel and (2) of the two lines *l* and *m* themselves. Let us consider this definition thoroughly, and see what it means. Two alternatives and only two are open: either the *l* and *m* by which *a* was determined intersect one another, or they do not intersect one another, although they are by hypothesis coplanar—that is, they are parallel. First, suppose that *l* and *m* intersect. Then there can be no lines parallel to both, for no line can be parallel to both of two intersecting lines in a space in which the lines we have defined have the properties we have attributed to them—that is, in a space in which our initial experience of the intersection of convex solids has the properties that we should naturally attribute to it. Consequently, all the lines which are either parallel or concurrent with both *l* and *m* must be concurrent with both *l* and *m*. For this reason, the generalized point *a* is made up of all the lines which, when taken together with *l* and *m* form concurrent triads—or in other words, which pass through the intersection of *l* and *m* but are distinct from either—together with the lines *l* and *m* themselves. *a,* that is, is the class of all the lines, in the sense in which we have already defined lines, which pass through what we should ordinarily call a certain point—namely, the intersection of *l* and *m.* That is, all the things that we wished to call generalized points remain generalized points when we replace our original definition of a generalized point by the one just given. There are, however, things that are generalized points in accordance with our new definition which would not be called generalized points under our old definition. If the *l* and the *m* by which our generalized point *a* is defined do not intersect, then since they are coplanar, they must be parallel. It is manifestly impossible for three lines, two of which are parallel, to be concurrent, so that all the lines *x* which are parallel or concurrent with *l* and *m* both must be parallel with *l* and *m,* and consequently the generalized point *a* reduces to *l* and *m* and all the lines parallel to them: that is, to all of those lines that we have defined which point in a single direction. These generalized points which consist in all the lines in the region accessible to our experience pointing in some direction and are generalized points by our second definition and not by our first tentative one we shall call *irregular* generalized points, for example, a direction may be regarded as an irregular generalized point. If our experience of the intersection of convex solids behaves as it should, every generalized point either corresponds to a point of ordinary geometrical space or is a direction.

It will be noticed that the definition of generalized points on which we finally agreed satisfies that criterion which we have insisted that every definition in this paper should fill—that it introduces no notions other than that of our experience of the intersection of convex solids or notions which have already been defined in terms of this. This is the case because it involves only the notions of a plane and of concurrence or parallelism, which have already been defined in the requisite manner. Here is perhaps as good a place as any to repeat at some little length the reason why we are always so insistent that our definition shall involve no other notion than that of our experience of the intersection of convex solids and notions already defined in terms thereof, except, of course, such purely abstract and logical notions as those of class, member, disjunction, etc. It will be remembered that our object in this course of lectures is to show that the geometrical properties of space are due, not to any peculiar formal properties of our original spatial experience, but to the method in which the points, lines, planes, distances, etc., of geometry are defined as constructions from or tabulations of our original spatial experience. Therefore, while it is perfectly in order for us to make use of the conventional geometrical notions of lines, points, planes, etc., and of the theorems of geometry concerning them that we may show that the things within our system of definitions have such not purely formal properties as we should naturally expect entities of their respective names to have, we must not introduce into our definitions any notions which we do not either explicitly start from or reach by some chain of definitions starting from these explicit primitive notions alone. Otherwise, we have not given an adequate characterization of the method by which we attain to our final geometrical concepts, and are hence not able to give a satisfactory proof that the fact that these notions obey the laws of geometry results simply and solely from the method by which they are reached and defined in terms of the notions which we have explicitly taken as primitive. It is for this reason that we do not make use, for example, of the first definition of generalized points which we gave; we had in our possession no criterion of the intersection of two lines or of the concurrence of three lines which was independent of our every-day geometrical notions and depended solely on our experience of the intersection of convex solids, which we have taken as primitive, so that our first definition of a generalized point, which was essentially dependent on the notions of intersection and of concurrence, failed to satisfy our demand that the purely geometrical properties of the entities defined by it should result wholly from the method in accordance with which they were derived from our fundamental experience.

Our final definition of generalized points is, then, an adequate definition, and covers all the entities which one would naturally call points, but it still suffers from a defect which, it is true, is essentially different from the one that constituted an objection to our very first definition of points, which this new definition of generalized points was designed to replace. Our first definition of points was defective in that it did not give us enough points—instead of yielding us all points in space, it only yielded us all those points in a certain region of space—while our definition of generalized points has the disadvantage of covering, not only such entities as our good ordinary every-day points, but also such entities as directions, which would not be termed in points in ordinary Euclidean geometry. The problems then arise, how can we distinguish our good, ordinary, every-day points from the things which we have called directions or irregular generalized points? and, can this be done by finding some property, possessed by all directions, not possessed by any other generalized points, and definable in terrms of our experience of the inetrsection of convex solids alone? We shall discuss both these questions in the next lecture and give reason why we should answer the second of them in the affirmative. We shall thereby have given a definition of a point which, to say the least, apply to a set of entities correlated in a one-one manner with the points of ordinary space (providing that the experience of the intersection of convex solids has those properties which we should naturally expect it to have), which might possibly be said to *be* the set of all the points of ordinary space, seeing that we do not yet know what the points of ordinary space are. This set is the set of all generalized points, with the irregular generalized points removed from it. The definition of these points will not, however, be that which we shall accept as final in this course of lectures, for this definition of "point" will not secure to space its ordinary geometrical properties, whatever the formal attributes of our experience of the intersection of convex solids may be, whereas we desire to arrive ultimately at definitions of points, lines, distances, etc., which will of themselves secure to space its geometrical properties, and will entail as consequences the usual theorems of geometry.

Another thing to which I desire to call attention again in this place is the essential difference between the things which we called points in accordance with our first definition of that term and generalized points. The class of all our generalized points is not made up simply of all the points, in the first sense in which we defined points, together with certain additional entities; all our generalized points are classes of the lines that we have already defined and all such lines are classes of points, in the first sense we gave to that term. Consequently, none of these latter points can any more be generalized points than a collection of collections of tables can be a table, or a group of families of human beings can be a human being. Nevertheless, to a point in our first sense, there may happen to correspond a certain generalized point. If our experience of the intersection of convex solids has the properties that it would naturally be expected to have, the class of all the lines, in the sense in which we have already defined lines, which contain as a member a certain point, in our first sense, constitutes a generalized point which, we may say, corresponds to or represents the point, in our first sense. It is not by any means true, however, that, whatever formal properties our initial experience of the intersection of convex solids might have, it would follow *merely from the definitions of points, lines, and generalized points in terms of our initial experience* that the class of all the lines passing through some point, in our first sense, is a generalized point. Now, we desire to give a definition of the generalized points corresponding to given points, in our first sense, which, while it will yield the same result as the definition just suggested in the case where our fundamental experience behaves itself, and will make the generalized point consisting of all the lines that we have defined which contain a given point, in our first sense, correspond to this point, will not be dependent for its importance, to all intents and purposes, on the hypothesis that our initial experience of intersection has not been misbranded and possesses all those properties which we should naturally associate with its name. This result we secure by saying that if a generalized point contains two distinct lines as members, while these lines themselves both contain a certain point of our first sort as members, the generalized point shall correspond to the latter point. If points in our first sense are analogues of those of the points of an ordinary geometrical space which lie within a certain closed convex region, as they are when our initial experience behaves itself, and generalized points are analogues of pencils of all the linear segments intercepted by the surface of this region which are concurrent or parallel with a given coplanar pair of such segments, including the linear segments belonging to this pair, then, since if any such pencil contains two segments both containing a given point it contains all of our segments that pass through this point, any generalized point which contains two distinct lines passing through a given point, in our first sense, will consist of all of those of our lines that pass through this point. If our initial experience does not behave itself, however, and has other formal properties than those which we have attributed to it in this and previous lectures, it will be a much more common thing for a generalized point to contain two lines passing through a given point than for it to contain all the lines passing through that point, so that our new definition of the correspondence between a point, in our first sense, and a generalized point will be much more frequently applicable than the one which we have rejected. It should be noticed that our new definition of this relation of correspondence offers us no security that to each of the points, in our first sense, there shall correspond one point and one only, nor that the converse correspondence is unequivocal. To one generalized point there may correspond many points in our first sense, and to one point in our first sense there may correspond many generalized points. We can define unequivocally, however, the *class* of generalized points corresponding to a given class of points in our first sense. This class is made up of all the generalized points which correspond to any member of the class of points in our first sense, whether this correspondence is one-one or not. If we possess a definition of the class of all the points, in the first sense, in some region, we may define in this manner the class of all the generalized points in the same region.

We have completed, then, our discussion of generalized points. To be able to make any use of generalized points, however, we must be in a position to assert geometrical statements concerning them. The first and most important of all geometrical statements are those which concern the collinearity of points or the concurrence of lines. To be able to make such statements we must know what a line is and when three points are collinear. Since our generalized points are classes of lines, in a sense of the term which we have already defined, such that all the lines belonging to a given generalized point are either parallel or concurrent, and since, when these lines are concurrent, the ordinary geometrical point to which our generalized point corresponds is the apex of the pencil of lines constituting the generalized point, it might be thought that a necessary and sufficient condition for the collinearity of three generalized points would be that they should contain in common a member—i. e., a line in our previously defined sense. One should bear in mind, however, that the things which we called lines in accordance with our former definition did not correspond to the complete lines of ordinary geometry, but rather to the linear segments intercepted by the surface of a certain closed convex region of space to which we shall refer briefly as R, whereas there are generalized points representing all the points in space. Since R is closed, there are lines in space which do not intersect R. Let *l* be one of these, and let A, B, and C be three distinct points on *l*. Since, to put it crudely, a generalized point is made up of all the linear segments intercepted by the surface of R which are aimed at some point or in some given direction, if we should adopt the definition of collinearity which would make three distinct generalized points collinear when and only when they all contain some member (i. e., some line in the sense already defined) in common, A, B, and C cannot be collinear, for there is no segment of a line intercepted by the surface of R that points at all three of them. We must therefore search about for another definition of collinearity.

The definition of collinearity that we have just rejected is, however, a sufficient condition of collinearity and an adequate preliminary definition of it, provided that the three generalized points A, B, and C, whose collinearity we assert, are on a line which passes through the region R. It is fairly obvious, too, that if A, B, and C be any three generalized points, if *l* is a member of A, *m* is a member of B, and *n* is a member of C, and if *l*, *m*, and *n* are coplanar, in the sense in which we defined the coplanarity of our lines in our last lecture, we should naturally say that A, B, and C are all in the plane of *l*, *m*, and *n*. Furthermore, if R is any solid region in space and *l* is any set of points in space, it is obvious that if at least two planes, *p* and *q*, can be contructed in such a manner that each contains *l* and each possesses a planar area in common with R, *l* is made up solely of the points in some line. These two planes have only the points of the single line *l* in common. A, B, and C, then, are collinear, if they both belong to two distinct planes passing through R in planar regions. That is, by virtue of the fact which we stated just a second or so ago, if *l* and *l'* are members of A, if *m* and *m'* are members of B, and if *n* and *n'* are members of C, while *l*, *m*, and *n*, on the one hand, and *l'*, *m'*, and *n'*, on the other, form triads of coplanar lines, A, B, and C are collinear, provided that the plane of *l*, *m*, and *n* is different from that of *l'*, *m'*, and *n'*. This latter condition may be formulated as follows: there shall not be any two distinct intersecting lines of our first sort meeting at a point *a* of our first sort which are each cut in a point other than *a* by every one of the six lines, *l*, *m*, *n*, *l'*, *m'*, and *n'*. On account of this fact and on account of the fact that the coplanarity of the lines of our first sort can be defined solely in terms of our experience of the intersection of convex solids, the necessary condition which we have given for the collinearity of three generalized points is formulable purely in terms of notions that can themselves be defined ultimately in terms of our experience of the intersection of convex solids alone. To show that we can regard this condition as a definition of the collinearity of three generalized points, it simply remains for us to prove that this criterion is also a sufficient criterion of the collinearity of three of those generalized points that correspond to ordinary geometrical points, and to investigate what is the natural interpretation in the language of ordinary geometry of the relation of collinearity thus defined when it relates directions to one another or to ordinary generalized points. Then we may be sure that the relation which we have defined as collinearity will have the properties one would naturally associate with its name when it connects ordinary generalized points, and we shall know the translation into ordinary geometrical language of the relation we call collinearity, when it holds among directions or between them and other generalized points. We shall finally close this chapter with a definition of "generalized lines"—i. e., lines made up of generalized points—in terms of the relation of collinearity.

Our first remaining task is, as we have said, to show that the condition we just gave for the collinearity of three points is a sufficient one. We wish to prove, that is, that if A, B, and C are three generalized points which we should naturally call collinear, then six lines in our first sense of the term—namely, l, l', m, m', n, and n'—can be assigned in such a manner that l and l' shall belong to A, m and m' to B, and n and n' to C, while l, m, and n, on the one hand, and l', m', and n', on the other, form coplanar triads of lines in distinct planes. Returning to the geometrical interpretation of the lines in our first sense, we see that this condition will be fulfilled if it is always possible, given a convex solid region in space, and a set of three collinear points anywhere in space, to draw six lines, two passing through each of the points in question, in such a manner that all pass through the solid region, and that there are two distinct planes, each containing one line through each point. This is practically equivalent to the obvious proposition that two distinct planes can be drawn, passing through any line you please, and cutting a given solid region of space in planar areas. Consequently, if our experience of the intersection of convex solids lives up to its name, the relation which we have defined as the collinearity of generalized points, in so far as it applies to generalized points other than directions, is at least the precise analogue of the relation known by the same name in ordinary geometry. All that remains now is to consider the properties of the relation we have defined as collinearity when it relates irregular generalized points to one another or to ordinary generalized points.

There seem, at first sight, to be three possible ways in which directions may enter into relations of collinearity. It seems *a priori* possible that either (1) three directions should be collinear, or that (2) two directions and one ordinary generalized points should be collinear, or that (3) one irregular and two ordinary generalized points should be collinear. However, we shall see that, provided our initial relation of apparent intersection behaves itself, the second alternative just suggested cannot occur. For, if it does occur, let us suppose that the two directions in question are A and B, and that the ordinary generalized point is C. Then, by the definition of collinearity, there are two distinct planes, each containing a member of A, a member of B, and a member of C. Since A and B are each sets of parallel lines, and represent different directions in space, the two planes in question are parallel, for it is an elementary geometrical theorem that two distinct planes such that one of them contains two given lines in different directions and the other of them contains two lines parallel, respectively, to these lines are parallel. Since, however, each of the two planes contains a member of C, and since C, being an ordinary generalized point, is made up of segments of intersecting lines, the two planes in question intersect. We thus obtain a flat contradiction in the case where two of three collinear generalized points are directions and the third is not, and where our initial experience has the formal properties that are attributed to it in ordinary geometry.

Let us, then, consider the first of the two remaining alternatives. What are the conditions under which three directions will be collinear in accordance with our definition? Let these irregular generalized points be A, B, and C. By our definition, they will be collinear if there are two distinct planes such that each of our three points contains a line in each of the planes as a member. By what we said in the last paragraph, these two planes must be parallel. Three directions, that is, are collinear when and only when each of them possesses as a member a line, in the sense already defined, in each of two given parallel planes. Since, however, a direction is made up of all the lines, in our first sense, parallel to a given line of that sort, and since if two planes are parallel each of them contains some line parallel to any line you please of the other, the condition that we have just given for the collinearity of three directions is equivalent, provided that our points, lines, etc., have the ordinary geometrical properties, to the condition that some member of A, some member of B, and some member of C be coplanar, or in other words, since we should naturally say that a point lies in the same plane as a line through it—that is, as one of its members—it is equivalent to the condition that A, B, and C all belong to some ordinary plane. To sum up, in a space that obeys the laws of geometry, the collinearity of directions is equivalent to coplanarity, in some ordinary geometrical plane.

The only remaining alternative is that the three points A, B, and C should consist of two ordinary generalized points and one direction. Let A and B be the ordinary generalized points, and let C be the direction. If A, B, and C are to be collinear, there must be two distinct planes, each of which contains a member of A, a member of B, and a member of C. We should naturally say that A and B are on the intersection of these two planes, since each is situated on (i. e., contains as a member) a line in each plane. Since the two members of C lying, respectively, in each of these planes are parallel, as C is made up of all those of our lines that point in some single direction, both of these lines must, to put it crudely, be parallel with the intersection of the two planes, the line AB, for it is a theorem of elementary solid geometry that if *p* and *q* be any two intersecting planes, if *l* is a line in *p*, and if *m* is a line in *q*, the only condition under which *l* and *m* can be parallel is that both should be parallel with the line of intersection of *p* and *q*, unless one of them coincides with this line, and the other is, of course, parallel with it. For suppose the contrary to be the case. Then the line of intersection of *p* and *q*, since, by the definition of l, it is coplanar with *l*, and since, by hypothesis, it neither is parallel to it nor coincides with it, must cut *l* in a single point. *m* and *l* are coplanar, since they are parallel, and consequently the intersection of *p* and *q*, which cuts them both, lies in the same plane as *l* and *m*. Therefore the plane which *l* determines together with this intersection—namely, *p*—is identical with the plane that *m* determines together with this intersection—that is, *q*, and since we assumed that *p* and *q* were *two* planes, and not a single plane, we get a contradiction. This proves our theorem. As a consequence, if space is decently behaved, two ordinary generalized points A and B are collinear with a direction C when and only when C may be said to be made up of lines parallel to the line AB. In this case we may very naturally call C the direction of AB.

We are now in a position to define generalized lines. The generalized line AB is the class of all generalized points that are collinear with A and B, together with the two generalized points A and B themselves. In a properly behaved space generalized lines, like generalized points may be divided into two classes: ordinary generalized lines and irregular generalized lines. An ordinary generalized line is made up of all the ordinary generalized points which we should usually regard as lying in that line, together with the irregular generalized point which is the direction of the line. As a consequence, two parallel ordinary generalized lines always intersect in an irregular generalized point. An irregular generalized line is made up of all the irregular generalized points on these ordinary generalized lines which, we should naturally say, lie in a certain plane. This one may readily show. Every generalized line, therefore, contains irregular generalized points, and since even parallel generalized lines intersect—in irregular generalized points—it is clear that we must sooner or later find a way of distinguishing irregular generalized points from ordinary generalized points, in order that we may remove the former from our system, and obtain a set of generalized points which corresponds completely to the set of all the points in ordinary geometrical space; and hence, since we are still unaware just what the set of all the points in our ordinary geometrical space is, by accomplishing the task whose necessity has just been indicated, we shall obtain a set of entities which may be said to *be* the set of all the points in our ordinary geometrical space. To do this with the introduction of few or no new sorts of experience is, as we have already said, the task of our next lecture.

We have succeeded, then, in defining all the points and lines of space, together with certain additional entities which we have called *irregular* points and lines—they really include the points at infinity and lines at infinity, respectively, of projective geometry—in terms indirectly of our experience of the intersection of convex solids, but directly in terms of the points in a given convex region of space and of the linear segments intercepted by the surface of this region. The method which we have used here in the solution of this latter problem is not original, but is due to Bonola, who makes use of it in an article that appeared about 1903 in the *Giornale di Matematice.* We have omitted the theorems which he proves there, however, and have only made use of his definitions and method, because it is only these that are relevant to the main purpose of our paper. The general notion of defining points as classes of lines dates back at least as far as Pasch, *Vorlesungen über der neueren Geometrie,* and has since been used by many mathematical writers, notably Schur and Whitehead.

V

Parallel Lines and their Empirical Basis

IN OUR previous lectures we have arrived at certain definitions which enable us to express the lines and points of our ordinary, every-day space, or at least entities that correspond to them in a one-one fashion, under a few quite natural hypotheses, in terms of our experience of the intersection of convex solids, where only such solids as come nearer to us than a certain maximum distance are experienced as intersecting at all. These definitions, moreover, yield us entities corresponding to *all* the points and lines of our every-day space. As we showed in our last two lectures, the reason that a definition of *all* the points and lines of space is such a necessary step in the definition of geometrical notions in terms of our experiences is that unless we have definitions of points and lines which will give us all the points and lines of space, an adequate definition of parallelism is impossible; and without an adequate definition of parallelism one cannot introduce measurement and such notions as depend on measurement (e. g., distance, angle, area, and volume) into geometry, notwithstanding the fact that the study of these is one of the most important portions of ordinary Euclidean geometry. Now that we have given definitions that cover all the points and lines of geometry, it might be thought that it would be a very easy task to define the relation between two lines which consists in their being parallel. Two lines, one would naturally say, are parallel if and only if they lie in the same plane, but have no point in common. This definition, however, is objectionable for a reason different from any which has led us to discard certain of the definitions which we have formerly made: the space of points and lines which we reached in our last lecture is not, it is true, too poor and niggardly in points and in lines, but—and this is almost as bad—it is too *rich* in points and in lines. It will be remembered that we showed in our last lecture how, under the hypothesis that our experience of the intersection of convex solids has such formal properties as the name we have given it suggests, the entities that we defined as "generalized points" will not, to put it crudely, all turn out to be the points of ordinary geometry, for to some of our generalized points there will correspond in our everyday space, not points, but such *directions* as east or north or up. We further showed how, under the same hypothesis any two of the things we defined as "generalized lines"—i. e., lines made up of generalized points, always intersect if they are in the same plane, even if they are parallel, since a generalized line contains its direction as if it were a point upon it, and two parallel lines always have the same direction—for we do not regard east and west, or up and down, or any other two opposite directions as distinct. Since coplanar generalized lines intersect, even if they are parallel, it is impossible to define two generalized lines as parallel when and only when they are coplanar and do not intersect. How, then, are we to define parallelism in terms solely of our experience of the intersection of convex solids, which we have chosen to regard as primitive in this course of lectures? Manifestly, the simplest way to do this is to find some intrinsic peculiarity of those generalized points which represent directions, so that we may recognize these intruders and remove them from our system. But before we can execute the sentence of death on these anarchistic invaders, we must find out which of our generalized points are the culprits and which are not. Remember, what we are trying to do is to define all geometrical notions in terms of our experience of the intersection of convex solids *alone*. Either the diagrams we draw on the blackboard and the references we make to ordinary Euclidean geometry are only aids to intuition, to enable us to picture to ourselves what we are doing, and to keep us from getting tangled up in the prolixity and dullness that are inseparable from any purely abstract chain of definitions, or else these diagrams and geometrical phrases and chains of reasoning are just so many assurances that, *if* the experience of the intersection of convex solids with which we have started has such properties as one would naturally attribute to an experience of that name, the entities that we have called points and lines will have many properties belonging to the lines and points of ordinary geometry, and might be regarded as actually identical with the points and lines of ordinary geometry, without any very extravagant alteration of the usual meanings of these terms. Except in such merely illustrative and secondary ways, no use whatever of geometrical theorems, notions, or figures is to be made in this course of lectures. It is consequently not possible to us to adduce the usual geometrical notion of parallel lines for the purpose of distinguishing ordinary generalized points from directions, in any such way as to render this distinction an integral part of that geometry which we are trying to build up from the beginning, on the basis of experience alone. Until we reach a definition of parallel lines dependent upon our experience of the intersection of convex solids and upon no other notion, or else explicitly augment our list of those experiences which we take as primitive in such a manner as to be able to give a completely adequate definition of parallel lines—and we have done neither of these things so far—we are not entitled to define a generalized point as a direction when and only when it is made up of parallel lines of our first sort, even though we thus defined directions in our last lecture. We have thus the problem before us: how are we to distinguish directions from other generalized points, either by means of a definition of the parallelism of the lines which we defined in our second lecture, which definition must be made entirely in terms of experience, or through some method not dependent on a previous definition of parallelism?

We shall develop in this lecture two alternative definitions of the undesirable irregular generalized points or directions. One of these definitions will involve no notion not already defined by us in terms of our experience of the intersection of convex solids, and will be to that extent superior to our other definition, which will demand the introduction of new ideas, entirely foreign to anything that has already been considered in this course of lectures. Notwithstanding this fact, the second method which we shall give for the definition of parallel lines will have certain marked advantages over the other one in that, for example, the notion which we define as parallelism by means of the latter is likely to depart further from our usual notion of parallelism than that which we obtain through the method that introduces notions not definable in terms of those already presented in this course. We shall first take up the method of definition of parallel lines which involves no new primitive experiences.

You all remember how our experience of the intersection of convex solids practically reduced itself to the actual intersection of other convex solids. These other solids we called a-solids, and we regarded the a-solid corresponding to a given convex solid as made up of the convex solid itself, together with what we called its *aura*—a certain layer surrounding it in such a manner that one a-solid is experienced as intersecting another when and only when the aurae of the two actually intersect one another. On several occasions we made the more or less natural assumption, which, it is true, was nowhere *absolutely* essential to our reaoning, that the aurae of all convex solids, and all parts of the aura of any convex solid, are of the same uniform thickness throughout. Let us suppose in all that now follows that this assumption is correct, and that the uniform thickness of all the aurae of convex solids is t. This being the case, it is obvious that no a-solid can be smaller or equal to a sphere of radius t. It is further obvious that a-solids can be found which approximate as closely as you please in shape and size to spheres of radius t, supposing that t exists, since convex solids may be constructed as small, and consequently as near to a point in size, as you please, whereas these spheres of radius t may be regarded as if they were formed by the adjunction to a point of an aura of thickness t. Now, I wish to define the class of all the points, in the first sense in which we used that term, which lie inside or on the surface of any of these spheres of radius t in terms of our experience of the interection of convex solids and of notions already derived from this experience alone. How am I to go about it, on the basis of what I have already said concerning such spheres?

Let it be remembered that when we defined a point as a class of convex solids, we so defined it that every convex solid which is experienced as passing through a point is a member of that point. As a consequence, the class of points which lie in the a-solid of a given convex solid—i. e., the class of all those points which are experienced as if they belonged to the convex solid, for the aura of a convex solid, which transforms it into an a-solid is that part of space in which it is experienced as lying, but does not lie—this class, I say, is precisely the class of all the points of which the convex solid is a member. It is a consequence of what we said in the last paragraph that if we take a sufficiently small convex solid, the class of all the points contained in its a-solid will approach as closely as we wish to the class of points in some sphere of radius *t*—or, as we shall say, to the class of all the points in some *a-sphere*. On the other hand, an a-sphere is always smaller than an a-solid. Since we can construct a-solids approximating as closely as we wish to any given a-sphere, though they never become quite as small as the latter, it is possible to determine any given a-sphere by a sequence of a-solids approximating to it. By a judicious choice of the members of such a sequence, it is possible to make each approximation to a given a-sphere surround the next more accurate one. In such a case, those points which belong to the a-sphere will be precisely those which lie within *every one* of the a-solids belonging to the sequence. We are consequently in a position to give a definition of an a-sphere, *qua* set of points, in our first sense, without introducing any other notions than such as are definable in terms of our experience of the intersection of convex solids, provided we are able to give a definition of a sequence of a-solids leading to an a-sphere. It is obvious that a partial list of these properties which a sequence leading to an a-sphere must have includes (1) the property which consists in there being no a-solid contained in every member of the sequence (for then the members of the sequence would not approach as near as you please to an a-sphere, but rather to an a-solid), and (2) the property which consists in the fact that each member of the sequence is contained in the previous one. The phrase "is contained in" which is used in the above statements has the following interpretation: one a-solid is contained in another when and only when all those points which lie in the first of the a-solids in the manner defined above also lie in the second. Using the phrase, "is contained in," in this sense, what are the various sorts of sequences of a solids which are such that (1) there is no a-solid contained in every member of the sequence, and (2) each member of the sequence is contained in the previous one? A little reflection will convince us that such sequences of a-solids approach and contain as the set of all the points common to all their members either (1) the set of all the points in an a-sphere, or (2) a cylinder of radius t, with hemispherical caps at its ends, or (3) a solid consisting of all the points whose distance from a given bit of a plane is not greater than t. It is easy to see that solids of classes (2) and (3) contain a-spheres, whereas no a-sphere contains another. We may therefore *define* a set of points, in our first sense, as the set of all the points in some a-sphere, when and only when it is made up of all the points common to all the members of a sequence of a-solids such that each term of the sequence contains the next, though there is no a-solid contained in every member of the sequence, provided that our set of points contains as a part no other set satisfying the conditions just formulated. To avoid certain exceptional possibilities which may arise when the a-spheres which we have just defined lie on the boundary of the region within which we experience the intersection of convex solids, we shall recast our definition so as to exclude all such sets of points as contain either but a single point or no point at all from the sets of points which we call a-spheres. If you follow this definition of an a-sphere step by step, you will see that no notion is involved in it other than that of our experience of the intersection of convex solids and notions which have already been defined in terms of this by various logical artifices.

Now that we have defined our a-spheres, how are we to make use of them in attaining the true goal of this chapter—the definition of the irregular generalized points? We are still a long way off from the solution of this latter question. Let it be noted, however, that we have defined, practically speaking, the class of all those spheres with radius t which lie in a certain region of space. Furthermore, any two equal spheres determine uniquely a right circular cylinder into which they both fit as a marble fits into a paper cylinder wrapped tightly about it, while any two lines lying wholly within the surface of such a cylinder are both parallel to the axis of this cylinder, and consequently to one another. Consequently, a generalized point will be a direction whenever it contains two members —i. e., two lines, in our first sense of the term—which both belong to a single cylindrical surface of the sort just indicated. It seems further likely that we can find a cylinder determined by two a-spheres pointing in any desired direction. Under this hypothesis, we may regard a generalized point as a direction when and only when it contains as members two distinct lines both situated within the surface of some cylinder determined by two a-spheres. As a consequence, if we are able to give a general definition of what it means to say that two lines, in our first sense, both lie within the surface of some cylinder determined by two a-spheres, we possess a completely adequate criterion by which we may distinguish directions from ordinary generalized points. What remains for us now is the definition of the surface of a cylinder determined by two a-spheres in terms of notions which we have already carried back to experience.

In our second lecture, we gave a definition of a linear segment made up of points, in our first sense of that term, and of the end points of such a segment. We shall say that a point p is *properly between* two other points, q and r, if p is distinct from both q and r, but lies on a linear segment of which they are the end-points. If A and B are two classes of points, we shall say that the region between A and B is the class of all of those points between some member of A and some member of B. Since a-spheres are so defined as classes of points as to include the points on their surfaces, the region between two a-spheres is the class of all points inside (not on the surface of) the two spheres, together with the class of all points *inside or on the surface of* the right circular cylinder stretching from a great circle of one a-sphere to a great circle of the other. Now, we wish to be able to define the surface of this cylinder, and, in general, the surface of the region between any two classes of points. But what intrinsic difference is there between the surface of a region and that part of it which is not surface? One would naively say that the surface of a region is that part of it which one can reach without digging down into it. Let us analyse this notion and see what the essence of it really is. To say that one can only reach a certain point of a region by digging down into it really means, to put it crudely, that if we probe into the region with some straight thing, such as a wire, the wire must touch the region some time *before* it reaches the point in question. Stated in precise language, this reads: if a linear segment contains a point in a region, but not on its surface, it contains at least one other point in the region besides. Since the surface of the region consists of those points of it which we can reach *without* digging down into the region, a strict definition of the surface of a region reads: the surface of a region R is the class of all those points which belong to R and also belong to some linear segment, taken as including its end-points, which has no other point in common with R. It will be found that this definition is in complete accord with our usual notions of the properties which the surface of a region should have. Let it be noticed, too, that we have so framed our definitions of the region between two solids that the surface of the region between two a-spheres consists only of the portion of a right circular cylinder lying between a great circle on one a-sphere and a great circle on the other, and does not contain the surface of the two projecting hemispherical caps which must be adjoined to this cylinder to make the complete region between the two a-spheres—for the hemispherical caps do not include their outer surface, as no point of this outer surface is situated between some point of one a-sphere and some point of the other, so that any linear segment which contains a single point of one of these caps contains other points of the cap as well.

Now that we have arrived at the definition of the surface of a cylinder stretching from a great circle of one a-sphere to a great circle of another, we are in a position to define a line lying in this cylinder, or, as mathematicians say, an *element* of this cylinder. We shall say that a straight line, in our first sense, is an element of one of the cylinders which we have just defined if and only if it contains a linear segment in common with the cylinder, for then it is obviously a prolongation of a piece of line lying entirely within the cylinder. As we have already said, all that remains for us to do if we desire a definition of irregular generalized points in terms of our experience of the intersection of convex solids is to define a direction as a generalized point which contains two distinct elements of some one of the cylinders that we have defined in terms of a-spheres. We can then go on and define two *generalized* lines as parallel when and only when they both contain some particular direction, or irregular generalized point, in the sense in which we have just now defined such entities. Although this definition of the parallelism of two generalized lines is made in terms of directions, which are themselves defined in terms of the parallel elements of a cylinder determined by two a-spheres, we cannot eliminate the intermediate step of defining directions, first, on account of the fact that the elements of a cylinder such as we have described are lines of our first sort, while we desire to define the parallelism of generalized lines, and secondly, because only such lines as are not further from one another than twice the thickness of the aural layer of a convex solid can both be elements of one of our cylinders, even though they be generalized lines.

Another point to notice concerning the definition of parallelism that we have just given is that the a-spheres which play so large a part in it are really nothing but the *minima sensibilia* of the philosophers, for an a-sphere is simply the region which a point seems to fill, to put it crudely. Of course, we do not mean to say that a-spheres are actually perceived; they may or may not be, but whichever is the case, they mark, like the *minima sensibilia* of Hume, some sort of a lower boundary of the perceivable, with respect to its spatial extension. The assumption which we have made that a-spheres are equal—that is, that the aural layer of all a-spheres and all parts of the aural layer of each a-sphere are of the same thickness—is, then, practically the same assumption as that which the philosophers of the English school made when they degraded the magnitude of a spatial region as determined by the number of *minima sensibilia* or "points" in it, which amounted to the same thing as supposing that all *minima sensibilia* were equal. Our formulation of this view, however, avoids many of the crudities of the orthodox Humian position. We have not avoided the difficulty, however, that it does not seem quite likely that the acuteness of our discrimination between bodies which do and bodies which do not intersect is the same throughout space. As a consequence, it would appear, contrary to what we have assumed, that the *minima sensibilia* remote from us and towards the boundary of that part of space which is open to our experience at all are larger than those more centrally situated, for the cruder our experience is, the larger must a thing be if it is to be perceived. This fact would make our "cylinders" determined by two a-spheres more or less conical in form, so that it would be possible for one of them to contain two non-parallel lines, or even not to contain any two parallel lines at all; and consequently our "cylinders" would be of no avail to us in the discrimination of irregular generalized points from ordinary generalized points. Therefore, though the definition of directions and that of parallelism which we have given define certain things and relations in terms of our experience of the intersection of convex solids alone, and in so far satisfy the requirements that must be satisfied by all our definitions, we have not any sufficient reason, to say the least, for supposing that the directions and parallelism that we have just defined represent precisely the directions and parallelism which we meet in the space of ordinary geometry. Since this is the case, an alternative definition of an irregular generalized point is desirable, as is also a definition of parallelism dependent thereon, and we shall try so to frame these new definitions that they shall not be entirely dependent for their applicability to the space of ordinary geometry on the assumption that all aurae of convex solids are of the same uniform thickness.

The first thing to notice is that any method which gives us three distinct directions that are not all directions of lines in any one plane—such directions as east, north, and up, for intance, satisfy this condition—that any method which does this, I say, yields us all the directions in space. We have already seen that any three directions lie in a single generalized line, in our every-day space, when and only when they are all, like east, north-east, and north, directions of lines in some single plane. This being the case, given any two distinct lines made up entirely of irregular generalized points and any particular irregular generalized point whatever, it is possible to construct a generalized line passing through the generalized point in question and cutting our two given irregular generalized lines in two distinct points; for, given any two non-parallel planes in ordinary space, and any direction whatever, one can construct a plane containing that direction—that is, parallel to any line pointing in that direction—and either cutting our two given planes in two distinct lines which point in different directions or else parallel to one of our two given planes, so that it contains every direction belonging to that one of our given planes. The reason why we have insisted that our two given planes shall not be parallel is that two parallel planes represent the same generalized line, for they contain the same directions. The facts which we have just stated, together with the fact that, in our ordinary space, only a direction can be collinear with two directions, entitle us to define the set of all irregular generalized points or directions in terms of any already known non-collinear triad of directions in the following manner: if A, B, and C are three known directions which are not all collinear, then a direction will be a generalized point which is collinear with a generalized point on AB and a distinct one on AC.

How are we to determine our A, B, and C, however?

Here we shall for the first time introduce into our definitions a notion not defined in terms of our experience of the intersection of convex solids. Let us suppose that we have somehow or other, by means into which we shall not now inquire, obtained the knowledge, not of the general notion of a sphere, but of four given sets of points, in our first sense, which we shall call unit spheres, whose centers are not all coplanar. What is required here can easily be given by experience, as it is not a general criterion of sphericity which we postulate, but an empirical acquaintance with a concrete set of entities. Let these four spheres be called X, Y, U, and V, respectively. The definitions of cylinders, their surfaces, and directions determined by them, which we gave in connection with our first theory of parallelism had no essential dependence on the fact that the spheres to which we applied them were a-spheres

—they can be applied without any alteration to the determination of an irregular generalized point in terms of any two equal spherical collections of points, in our first sense. As a consequence, since X, Y, U, and V, being all unit-spheres, are all equal, the cylinders XY, XU, and XV all determine irregular generalized points. These three points are not all collinear, as each is the direction of the axis of the corresponding cylinder, and these three axes are, by hypothesis, not all coplanar. As a consequence, we are able to define the class of all irregular generalized points in terms of the three directions, XY, XU, and XV, and therefore ultimately in terms of the four points X, Y, U, and V, and our experience of the intersection of convex solids, and of no other notion. We can now go on, as before, and define two lines as parallel when and only when they both contain a direction in common. This definition of parallelism does not, like our first definition, presuppose some such probably false assumption as that of the uniform thickness throughout all space of the aural layer of a convex solid, and moreover, as we shall see later, lends itself more readily than our first definition of parallelism to the introduction of metrical considerations into geometry, but it labors under the grave disadvantage that it does not depend exclusively on our experience of the intersection of convex solids, as our former definition does, but demands in addition the selection of four distinct convex solids from all those in the universe, not merely as spheres, but as the particular set of equal, non-coplanar spheres on which our whole subsequent theory of measurement will depend; while we are given in experience no four such spheres, singled out from among all convex solids. Thus neither of our two definitions of parallelism is entirely satisfactory. That a much better definition of parallelism than either, combining their advantages and eliminating their shortcomings, can be obtained without any very great difficulty, I do not doubt in the least, but I have not been able to formulate such a definition up to the present. You all see the philosophical interest of the solution of this problem: on it depends a satisfactory knowledge of what space really is, and of what the true meaning of geometrical propositions is, since many of the most important geometrical propositions deal either with parallelism, or with the measurement of distances, angles, etc., all of which depend in an ordinary Euclidean space upon parallelism.

To conclude: we have developed two distinct definitions of parallelism. Both of them start from the notion of the cylinder enwrapping two spheres. One of them involves ultimately no other concrete experience than that of the intersection of convex solids, but is not completely satisfactory, since, if we accept it, we must make an assumption concerning the nature of this experience which the experience probably fails to satisfy, in order to secure that the relation we call parallelism is really essentially the same as the relation usually known by that name. On the other hand, we have given a definition of parallelism which avoids this difficulty, but involves other experiences than that of the intersection of convex solids, namely, those of a certain set of four non-coplanar equal unit spheres. The present lecture covers the weakest portion of the theory which we are developing in this course, but there is every reason to believe that its weakness is only temporary, being due rather to the fact that the problems discussed have hitherto been little investigated than to the inherent nature of the subject.

In our next lecture we shall begin the investigation of how we are able to develop a theory of measurement on the basis of either of the definitions that we have given of parallel lines.

VI. The Logic of Distances

IN OUR last lecture, we arrived at two alternative definitions of parallel generalized lines. On the basis of either of these definitions we are able to define what is ordinarily called by mathematicians a vector. I can best illustrate what a vector is by introducing a few familiar examples. In a region so small that the sphericity of the earth is not noticeable, the relations, "ten feet to the north of," "a mile south-west of," "two yards up," etc., are, to all intents and purposes, vectors. A vector, that is, is the relation which holds between one point and another when they are separated by a given distance in a given direction. Let it be noticed, by the way, that a vector leading from A to B does not, in general, lead from B to A—for example, if A is ten feet east of B, then B is not ten feet east of A, but ten feet *west* of A.

One property of vectors in ordinary geometry—the property, indeed, which we shall use in defining them—is that if A and B are separated by a given vector, if C and D are separated by the same vector, and if the points, A, B, C, and D, do not all lie on a single line, then the line AC is parallel to the line BD. It is conversely true that if AC is parallel to BD, and if, further, AB is parallel to BC, the vector which separates A from B also separates C from D. These facts follow from the very elementary geometrical theorems that if a quadrilateral has a given pair of opposite sides equal and parallel, the other two sides are parallel, and that if each side of a quadrilateral is parallel to the opposite side, ech side is equal to the opposite side. From these facts and the fact that a vector always points in a given direction, it is easy to deduce the conclusion that to say that the vector that stretches from A to B is the same as the vector that stretches from C to D is precisely equivalent to the statement that AB is parallel to CD and that AC is parallel to BD, provided that A, B, C, and D are not all collinear. Furthermore, if A, B, C, and D are all collinear, and the vector from A to B is the same as the vector from C to D, then there must be some pair of points, E and F, which do not lie on AB, such that the vector from E to F is identical both with that from A to B and with that from C to D, as there obviously exist on any line pointing in a given direction all possible vectors in that direction. We may, therefore, *define* the vector separating A and B (which we shall write (AB)) as the relation which a point C bears to another point D when and only when either (1) AC is parallel with BD and AB is parallel with CD, or (2) A, B, C, and D are all on some single straight line, and there are two points, E and F, which are not on this line, and which are such that A, B, E, and F, on the one hand, and C, D, E, and F, on the other satisfy the condition that we laid down in one for A, B, C, and D: that is, if AB is parallel to EF, if AE is parallel to BF, and if CE is parallel to DF. Furthermore, a relation is defined as a vector in general if there are two points, A and B, such that the relation is the vector (AB). It will be noticed that this definition, like all those which have already been accepted by us, involves no notions which have not either been taken explicitly as primitive experiences by us, or else been defined in terms of these primitive experiences alone. When we have given, according to which of our definitions of parallelism we choose, either our experience of the intersection of convex solids alone, or this experience together with four selected solids which we choose as unit spheres, it is unambiguously determined whether any given thing is or is not a vector. I want you all to notice that we have not presupposed in any manner any notion already involving measurement in the definition of a vector which we have just given.

The next notion which we have to define is that of all vectors lying within a given region. Suppose that α is the class of all the points in a certain region of space. Let us consider only such vectors as separate point lying in α from points lying in α. Furthermore, let us consider even these vectors as relations which hold between points in α only. The relations thus obtained will be called the vectors-in-α. To give a concrete illustration of what I mean by the vectors-in-α, suppose that we give to our vectors the geographical interpretation which we gave to them in a previous illustration of the notion of vector, but that we limit our discussion to vectors on the surface of the earth—to vectors in such directions, namely, as are represented on a compass-card—excluding those, for instance, that poinit vertically upward, or 45° downward and north-east, etc. Let α be the class of all the points on the surface of some island in the ocean. We wish to consider only such vectors as lead from points on the island to points on the island. These vectors will be the class of vectors on the island, provided that we only consider them in so far as they join points on the island to one another. Suppose, for example, that our island is circular and one mile in radius. "Three miles to the north of" will not, then, determine a vector-in-the-island, for one will be unable to find any two points in the island such that one is three miles to the north of the other. The relation, "one mile to the north of," however, will determine a vector lying in the island, for it will be possible to select two points on the island such that one is one mile to the north of the other. The *vector-in-the-island* determined by the relation "one mile to the north of" will differ from the ordinary vector "one mile to the north of" in that, if A and B are not both points in the island, while A may be separated from B by the ordinary vector, "one mile to the north of"—i. e., A may be one mile to the north of B—it is impossible for A to be separated from B by the vector-in-the-island determined by the relation " one mile to the north of." This is the sole difference between the ordinary vector "one mile to the north of" and the vector-in-the-island known by the same name. The ordinary vector "one mile to the north of" will be called in our subsequent discussion the *extension* of the vector-in-the-island of the same name, and in general we shall call the ordinary vector R, from which the vector-in-α S is obtained by considering R only in so far as it relates members of α to members of α, the *extension* of S.

The next point which we shall take up is what it means to say that one vector R is *n* times as great as another vector S. Suppose that I go from a point A to a point B separated from A by the vector S. Suppose that from B, I go still further on to a point C, which is separated from B by the same vector S. Further, suppose that I might have gone directly from A to C by the vector R, or to put it a little differently, that A is separated from C by the vector R. Then we should naturally say that R is twice as great as S: that is, if by going *n* miles to the north from A to B, and by going *n* miles to the north from B to C, we find that we have gone *p* miles to the north in going from A to C, we should naturally say that *p* miles is twice as great a distance as *n* miles. In a precisely similar manner, we can define a vector R as 3, 4, 5, 6, ..., or *k* times as great as a vector S, if by taking 3, 4, 5, 6, ..., or *k* steps of the vector S we can sometimes take one step of the vector R. Let it be noticed that I only say that *sometimes* we can take one step of R by taking *k* steps of S, and not that this is always possible. In ordinary geometry, it is true that if one vector can sometimes be obtained by repeating another vector *k* times end on end, it can always be so obtained.

We do not wish, however, to have either the intrinsic adequacy or the generality of application of our definitions depend on the axioms or theorems of ordinary geometry, and it would be easy, by substituting other relations and entities than such as we should naturally call the relation of experienced intersection among convex solids or spheres, respectively, for our experience of the intersection of convex solids and for the four selected spheres in terms of which, by our second definition, we determined the class of all the irregular generalized points—it would be easy, I say, to obtain systems in which, for some values of A and B, one could go from A to B either by one R-step or by *k* S-steps, while for other values of A and B, it might be impossible to go from A to B by a single R-step, but still possible to go from A to B in *k* S-steps. Here, of course, we assume R and S to be vectors, in the sense in which we have already defined vectors, and since our definition of one vector as a multiple of another still applies, we shall say even in this case that S is one *k*th of R. We shall further define a multiple of a vector-in-a-region in a manner precisely parallel to that in which we have just defined a multiple of a general vector: that is, we shall say that if R and S are both vectors-in-α, and it is sometimes possible to accomplish a journey consisting of one step of R by *k* steps of S, we shall say that R is *k* times as great as S. We have thus given a definition of the relation which one vector-in-α bears to another *k* times as large, in terms only of our original experience of the intersection of convex solids, and perhaps of certain selected convex solids.

We shall now define what it means to say that one vector-in-a-given-region bears a ratio to another. A vector-in-α R bears the ratio m/n to another vector-in-α S if there is a vector-in-α T such that R is m times as large as T and such that S is n times as large as T. Thus, for example, the relation, "two miles to the north of," on an island is, by our definition, two-thirds as great as the relation, "three miles to the north of," on the same island, since there is a certain relation—namely, "one mile to the north of" on the same island—which is half as large as the former relation or vector and a third as large as the second. This definition now enables us to compare different vectors-in-a-region with respect to their magnitude. Be it noted, however, that these vectors which we compare must be vectors in the same direction, for it is impossible, for example, for us to go ten miles to the east by taking any number of successive one-mile steps to the north, nor is there any step which, when repeated a certain integral number of times, will give you a one-mile step to the north, and which, when repeated another integral number of times, will yield a ten-mile step to the east. Furthermore, we are not yet in a position to compare *any* two vectors in a given direction with one another with respect to magnitude, for two distances along one and the same line may be incommensurable. Suppose that I draw on this blackboard a line one foot long vertically upward from some selected point O, and another line of the same length running horizontally to the left from the same point. Connect the two free extremities of the linear segments so obtained by a linear segment *l*. Lay off a linear segment as long as *l* stretching upward from O. It is a simple matter to give a strict mathematical proof that there will be no distance which, when multiplied by some integer, will be equal to this latter distance, while it will equal one foot, when multiplied by some other integer. As a consequence, our definition of the ratio of two vectors will not enable us to ascertain what ratio our one-foot vector upwards from O bears to our vector in the same direction of length equal to *l*, notwithstanding the fact that they are vectors in the same direction. What we have just pointed out concerning vectors in a blackboard holds true, of course, for vectors in any region whatsoever: our criterion for determining the ratio of two vectors in the same direction does not enable us to compare the magnitudes of incommensurable ratios directly.

We wish, however, to be able to define the quotient of any vector—at least in a given region—by any other vector whatsoever. We wish to be able to regard a vector as a *distance* alone, regardless of its direction. We wish to find some way of comparing the magnitudes of incommensurable vectors. These things must be carried out on the basis of such experiences and notions alone as we have explicitly taken as primitive, if we are to obtain in our treatment of geometry any satisfactory theory of measurement. How are we to accomplish all this?

I shall first give a rough sketch of the method by which we shall obtain this desired result, and then I shall take this method up step by step and give precise definitions of everything I shall do. Roughly speaking, my method is to take some set of generalized points which may be regarded as a sphere, and to find a way of measuring every vector in space in terms of its diameter. As there is a diameter of our sphere in the same direction as any vector we please in space, one can see that the methods which we have already introduced for the comparison of the magnitudes of vectors in the same direction will be applicable in this instance. We shall consequently compare vectors in different directions by referring them to the diameter of our sphere as a standard, for that diameter is of a constant and determinate length, which is independent of the direction in which this diameter is taken. However, in our precise formulation of this method of comparing distances in different directions, there will appear no explicit reference to any diameters of our standard sphere. Our definitions will not depend for their intrinsic logical value as definitions upon any particular assumptions to be made concerning the set of generalized points which we take as our sphere—

we shall not even assume that it possesses anything such as one would naturally call a diameter. Such notions expressing peculiar properties of spheres as that of the center of a sphere or that of the radius of a sphere will likewise fail to make their appearance in the ensuing discussion.

The only spheres which we have already considered *qua* sets of points are sets of points in the first sense which we have given to the word "point" in this course of lectures. The spheres which we wish to consider now are sets of *generalized* points, for we wish to be able to consider our points as termini of vectors, and vectors, since they depend upon parallel lines, have been defined in the space of generalized points, and it was only in this space that we were able to arrive at the notion of parallel lines. We wish to avoid in our present lecture the necessity of supposing arbitrarily that any more sets of points are spheres than it is abolutely necessary for us to so consider. As a consequence of this, a definition of the sphere of generalized points corresponding to a sphere of points in our first sense is a desideratum. It will be remembered that we have already given a definition of the correspondence between a generalized point and a point, in our first sense, in terms of the experience of the intersection of convex solids alone. It will be further remembered that we had no reason to regard this correspondence as one-one except under the hypothesis that our experience of the intersection of convex solids has such properties as we should naturally attribute to it. Be this as it may, I wish to call attention to the fact, which is obvious upon a very slight reflection, that, given a set of points in our first sense, the set of all the generalized points which correspond to any of its members, whether univocally or not, is uniquely determined by the set of points in our first sense. By applying this fact to spheres, it is easy to see that, given any sphere of points in our first sense, there is always a certain single set of generalized points which we may define as "the same sphere" in the space of generalized points. This identification of these two spheres presupposes no other primitive notion than that of our experience of the intersection of convex solids.

Let us suppose that we have already singled out some sphere of points, in our first sense, and that we have obtained from it the corresponding sphere of generalized points. Let us discuss the class of all vectors-in-this-sphere—that is, by our definition, the class of all vectors in space, in so far as they connect points in this sphere. Since the sphere has a finite radius, if R is a vector-in-the-sphere, we cannot multiply R by a coefficient as large as we please, and yet have the resulting vector remain within the sphere. If we are on an island ten miles in diameter, it is clear that we can somehow manage to take one step of four miles to the north, or even two successive steps of four miles to the north, but it is impossible to make a journey in the island consisting of three consecutive steps of four miles to the north. For the same sort of reason as that which makes this latter task impossible—it involves a straight journey of twelve miles on an island only ten miles across—it is impossible, in general, to multiply a vector in a given spherical region by an arbitrarily large coefficient, and to obtain a vector in the same region as a result.

The question now arises, granted that it is true that a vector in a finite spherical region cannot be multiplied by an arbitrarily large numerical coefficient, just how is one to determine by how large a coefficient it may be multiplied, and still remain in the region under consideration? In particular, what relation does this coefficient bear to what one would ordinarily call the magnitude of the vector in question? The answer is fairly obvious if it is ordinary Euclidean geometry with which we are dealing, and if the "sphere" inside of which our vectors are confined is actually a sphere. It may be seen on inspection that, by our very definition of a vector, a vector not limited to a given region can be laid off on any line which points in the direction of the vector. Furthermore, not only can our vector always be laid off on such a line, but if *l* is such a line, and if A is any point on *l*, there is some point B on *l* which is separated from A by the vector in question. For example, if *l* is any line going north and south, and A is any point on this line, then if we choose any such vector as "five miles south of," there is a point on *l* which is separated by this vector from A—that is, which is five miles south of A. Furthermore, it is not only obvious that, as a consequence of this, every vector can be laid out on some line passing through the center of our sphere—for a line can be drawn in any desired direction through any point you please—but it is also true that every vector-in-our-sphere can be laid off on some diameter of the sphere. This is the same as saying, for instance, that if we are on a circular island, and can somewhere make a journey of ten miles to the north without leaving the island, we certainly can take such a journey along a diameter of the island. All this is the consequence of the familiar fact that a diameter is the longest linear segment which can be drawn within a circle or a sphere. From this principle it results that if R is a vector inside a given sphere, all vectors inside the sphere can be laid off on the diameter of the sphere pointing in the same direction as R. If the fraction which we have just defined as the ratio of two vectors be what we should naturally consider their ratio to be—and we have already shown that this is actually the case if our primitive notions live up to their names—it is obvious that what one would ordinarily call the ratio of the diameter of our sphere to any vector which we wish to measure is at least as great as the ratio of *any* vector on the diameter, in the appropriate direction to the vector which we wish to measure, and consequently, from what we have just seen, it is as great as the ratio borne to the vector which we desire to measure by *any vector whatsoever in the sphere.*

It is an easy matter to show that the notion of ratio which we have defined among the vectors in our sphere agrees precisely, as we have already said, with our usual notions of ratio, under the hypothesis that our fundamental experiences have not been misnamed. We are therefore justified in making use of the ordinary geometrical properties of ratio to find out what the magnitude of a vector-in-a-sphere has to do with the extent to which it can be multiplied by a ratio. Now, it is a commonplace of geometry that if we have given to us two linear segments, say AB and CD, either they are commensurable—that is to say, they bear a ratio to one another which is the quotient of one integer by another—or, given any distance *d,* no matter how small, it is possible to find on AB a point E, such that AE is commensurable with CD, while EB is less than *d.* The proof of this is at once simple and instructive, so perhaps I may be permitted to give it here. All of you know that when two quantities, such as distances, bear no ratio to one another of the form m/n, where m and n are integers, but are yet comparable with respect to their magnitudes, we say that the ratio of one quantity to the other is irrational, and we represent it by a non-terminating decimal fraction, such as 3.141592.., which is π, or the ratio of the circumference of a circle to its diameter. Now, what does it mean, for example, to say that the circumference of a circle is 3.141592..... times as large as its diameter? It means that if we take a distance 31/10 as large as the diameter of a circle, we shall fall less than 1/10 of the diameter short of the circumference, that if we take a distance 314/100 as large as the diameter, we shall fall less than 1/100 of the diameter short of the circumference, that if we take a distance 3141/1000 of the diameter, we shall fall less than 1/1000 of the diameter short of the circumference, and so on indefinitely We can thus define a distance which is commensurable with the diameter of a circle, and yet differs from its circumference by less than $1/10^k$ of its diameter, for any assigned positive integral value of k. It is obvious that we can make $1/10^k$ as small as we please by making k sufficiently large. By using precisely this line of reasoning, we may prove our thesis: that, given any two incommensurable linear segments, AB and CD, and any distance d as small as you please, there is a point E on the segment AB such that AE is commensurable with CD—that is, the vector AE bears a ratio, in the sense already defined in terms of our fundamental notions, to the vector CD—while the distance BE is less than d. As a consequence, it is possible to lay off on some diameter of our standard sphere vectors which are commensurable with any given vector in the sphere, say R, and which, if they do not completely fill up the diameter, leave a remainder which can be made smaller than any assignable vector. Therefore, it is natural for us to say that, although the ratios which other vectors bear to a given vector R can never exceed what we should properly call the ratio of the diameter of the sphere to the given vector R, there is no smaller ratio which is not exceeded by some of the ratios which other vectors in the sphere bear to R. That is, if we call the class of all the ratios which other vectors in the sphere bear to R by the name α, the ratio which the diameter of our sphere bears to our vector R will be larger than or as large as any member of α, yet there will be ratios which belong to α yet differ from this latter ratio which the diameter bears to R by less than ε, when ε is any assigned quantity, however small.

In the preceding paragraph, we came across a notion which is of the highest importance for the determination of the magnitude of a vector. This notion is that of a number *n* which is larger than or as large as all the members of a given class K of numbers, yet is such that there are members of K which differ from *n* in magnitude by as little as you please. It is obvious that there is only one value of *n* which can have this property, for a given class K, since if *m* also had the property in question, it would, of necessity, be either larger or smaller than *n*. If larger, since some members of K would differ from it by less than ε, however small ε might be, they would differ from it by less than *m-n*, and hence, contrary to our hypothesis, would be greater than *n*. If *m* is less than *n*, a precisely similar contradiction arises. As a result, the ratio which a vector R bears to the diameter of our sphere of measurement is uniquely determined by the class of all the ratios which other vectors in our sphere bear to it. Since this is the case, there is no objection to our *defining* the notion—previously undefined—of the ratio which the diameter of our sphere bears to the vector-in-the-sphere R as the number which is greater than all the ratios which vectors-in-the-sphere bear to R, but to which these latter ratios approach as near as you please. This definition will involve no notions not already incorporated into our system, and will be sufficient to secure the unicity of the ratio which the diameter of our sphere bears to our given vector R. If there are ratios which other vectors bear to R and these are all less than some fixed finite number, it is possible to prove by means of our definition alone that there is some number which represents the ratio which the diameter of our sphere bears to our given vector, although this proof involves certain elementary mathematical considerations, drawn from the modern theory of aggregates, which would be a little too intricate for us to consider here.

In ordinary Euclidean space, unless R is a vector that connects a point to itself, all the ratios which other vectors-in-our-sphere bear to R will be less than some fixed finite number, so that the ratio which the diameter of our sphere bears to R is actually determined as some particular number, if there are any vectors in our sphere that bear ratios to R. Now, it follows from our definitions alone that R bears the ratio 1 to itself, so that there are always vectors which bear the ratio 1, at least, to R. From this it follows, first, that every vector which cannot be repeated an indefinite number of times end on end and still remain within our sphere of measurement determines some single number that represents the ratio of the diameter of the sphere to R, and second, that this latter number is always greater than or equal to one, since it is greater than or equal to any ratio which R bears to another vector-in-the-sphere, and one of these ratios is equal to one. Where a vector R can be repeated end on end an indefinite number of times, and there are vectors smaller than the diameter of our sphere—i. e., vectors-in-our-sphere—that bear as large a ratio as you please to R, the natural thing to say would seem to be that the ratio of the diameter of our sphere to R is infinite. In a space living up to the axioms and theorems of ordinary geometry, if our sphere is a genuine sphere, this latter situation can only arise when R is the vector that connects a point with itself. It is obvious that since a step of this vector leaves you just where you were, any number of consecutive steps of this vector will likewise leave you just where you were, and will not take you outside of our sphere of measurement.

Let us consider for a while the ratio which a given vector-in-our-sphere bears to the diameter of our sphere. Every vector-in-our-sphere bears to its diameter a ratio which may naturally be regarded as the reciprocal of the ratio which the diameter of our sphere bears to the vector in question, and which we may *define* as this reciprocal as the notion of this ratio has not been already defined. In this context, we shall regard zero as the reciprocal of ∞. We shall, for short, call the ratio which the vector-in-our-sphere R bears to the diameter of this sphere the *index* of R with respect to the sphere of measurement in question. We may naturally regard the index of R in a certain sphere as the expression of the magnitude of R in terms of the diameter of our standard sphere as a unit. It follows from the fact that the ratio of the diameter of our standard sphere to any vector-in-the-sphere cannot be less than one, that the index of any vector-in-our-sphere, being the reciprocal of the ratio which the diameter bears to it, cannot be greater than one. This is as we should expect—no vector-in-a-sphere can exceed in magnitude the diameter of the sphere. Further, the index of a vector need not be a rational number, as may be seen without much difficulty by a mathematical analysis of the stages through which we have gone in defining it, for its reciprocal, the ratio which the diameter of our standard sphere bears to the vector-in-the sphere in question is defined, not as a ratio, but as a limit of ratios, and a limit of ratios may be an irrational number. The importance of this fact arises from the fact that it makes it possible for us to measure by their indices not only such vectors as are commensurable with the diameter of our sphere of measurement, but also vectors which are incommensurable with this diameter. This measurement of incommensurable distances by a common unit would have been impossible if we had tried to found a theory of measurement directly upon the notion of ratios among vectors, without the introduction of our standard sphere. Another fact which makes us introduce the standard sphere into our theory of measurement is that it enables us to compare vectors in different directions, as we have already seen, and makes it possible for us to consider them as mere distances, and not as vectors—i. e., *directed* distances. The index of a vector represents it in terms of the diameter of the sphere in its direction as a unit. However, the length of one diameter of a sphere is the same as the length of another diameter of the same sphere, no matter in what directions the two diameters of the sphere point. As a consequence of this, the index of a vector expresses its length in terms of a unit which is independent of the direction of the vector. We are thus enabled to say that a vector-in-our-sphere in one direction is as long as, or half as long as, or twice as long as, etc., a vector in another direction, according to whether the index of the first vector is equal to, or half as great as, or twice as great as the index of the second, respectively.

To conclude, we have defined in this lecture the notions of a vector, and of a vector-in-a-region. We saw that these could be regarded as magnitudes, and we defined ratios among them in terms solely of notions that we have already taken as fundamental—that is, in terms only of the experience of the intersection of convex solids, or in terms of this, together with our experience of four selected spheres, according to the one of our two alternative definitions of parallelism that we choose. Then, given any region whatever, we saw how we could define a system of measurement in terms of it which would always enable us to measure all the vectors-inside-the-region in terms of a common unit, and which, in case our space should satisfy the axioms of ordinary Euclidean geometry, and in case our selected standard region should be an ordinary sphere, would give us an ordinary, every-day Euclidean system of measurement. In our next lecture we shall consider how this theory of measurement may be extended so as to cover, not only all distances within a certain sphere, but all distances in space, and we shall consider on what principle our standard sphere of measurement shall be selected.

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